Topological Limit Shape transitions Melting of Arctic Circles

Dimitri Gangardt,

James Pallister

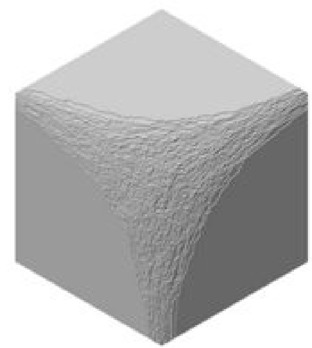
Alexander Abanov

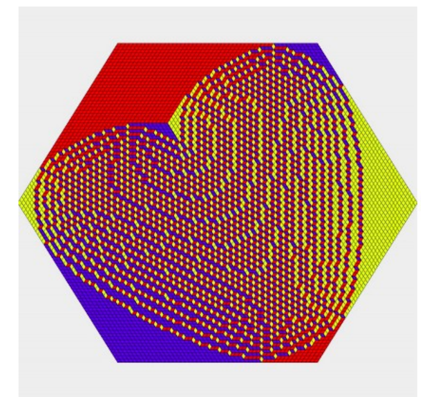
James S Pallister, DMG, Alexander G Abanov, J. Phys. A: Math. Theor. 55 304001(2022)

Երևան 23 June 2023

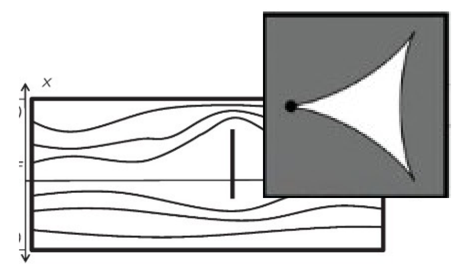
Limit Shape in Statistical Physics

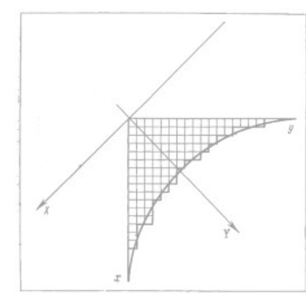
Formation of a non-random shape in thermodynamic limit of random/statistical systems.





- Equilibrium Shapes of Crystals
- Random Tilings
- Directed Polymers
- Random Young Tableaux



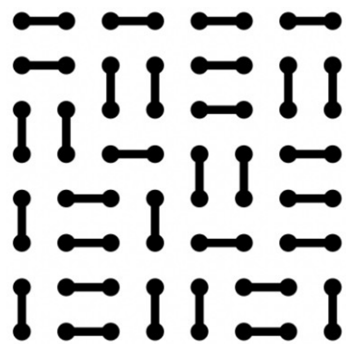


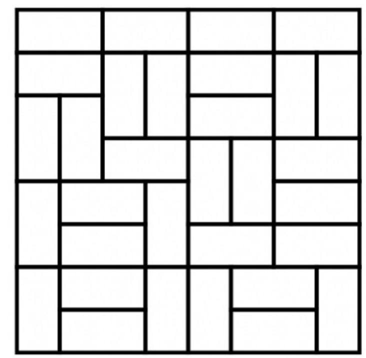
Vershik, Kerov, '77; Pokrovsky, Talapov, '78, '79; Elkies, Kuperberg, Larsen, Propp, '92; Jockusch, Propp, '98; Prahofer, Spohn; Johansson; Borodin, Gorin; Nienhuis, Hilhorst, Blote; Cohn, Kenyon, Propp; Kenyon, Okounkov; Abanov; Kenyon, Okounkov, Sheffield; Reshetikhin; Allegra, Dubail, Stephan, Viti; Colomo, Pronko, Zinn-Justin, Sportiello; Adler, Johansson, van Moerbeke; Corwin, ...

Domino tiling

<u>Problem</u>: In how many ways one could tile the 8 x 8 chessboard by dominos of the size 2 x 1?

Kasteleyn, 1963





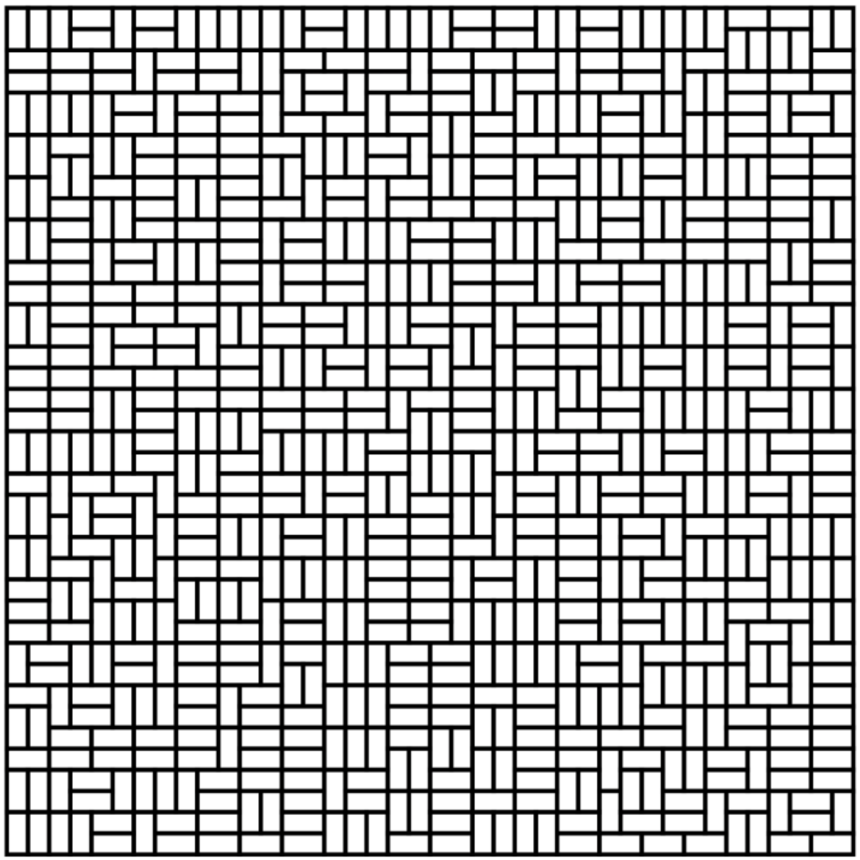
$$Z_{n\times m} = \prod_{j=1}^{[\frac{n}{2}]} \prod_{k=1}^{[\frac{m}{2}]} \left[4\cos^2 \frac{\pi j}{n+1} + 4\cos^2 \frac{\pi k}{m+1} \right]$$

$Z_{8\times 8} =$

	7.06418...	×	5.87939...	×	4.53209...	×	3.65270...
×	5.87939...	×	4.69459...	×	3.34730...	×	2.46791...
×	4.53209...	×	3.34730...	×	2	×	1.12061...
×	3.65270...	×	2.46791...	×	1.12061...	×	0.24123...

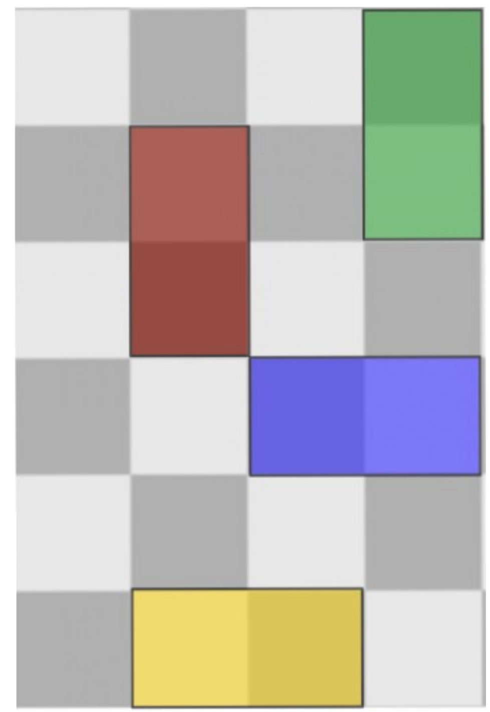
$= 12\,988\,816$

Colored dominoes



Random tiling of 40x40 lattice
(totally about 10^{197} tiling configrations)

Let’s colour dominoes of different orientation and different check board content

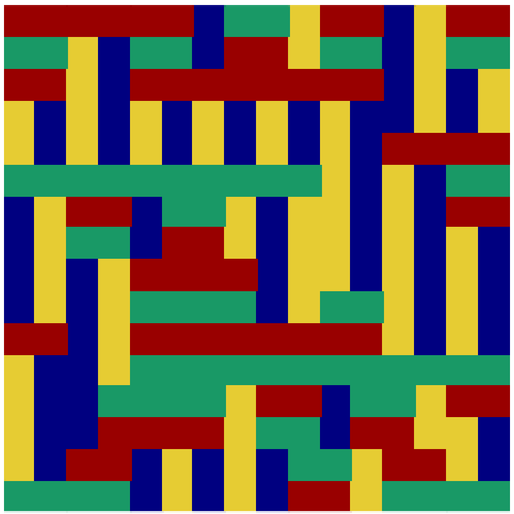


Երևան 23 June 2023

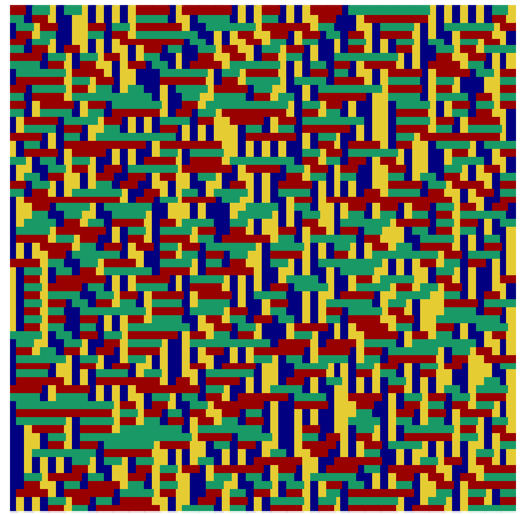
Spoiler: can be mapped onto a free fermionic model

Coloured tilings

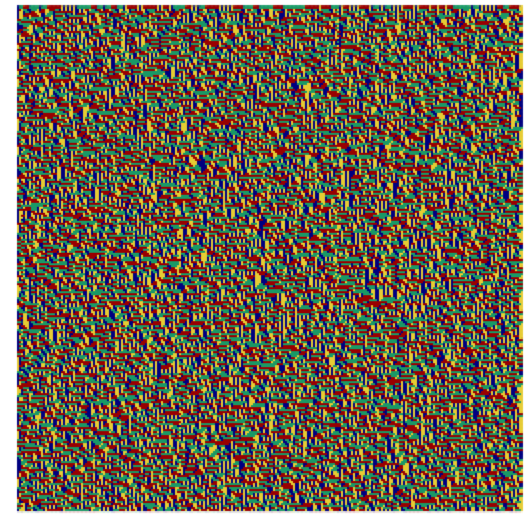
Stephane 2020



$L = 16$



$L = 64$



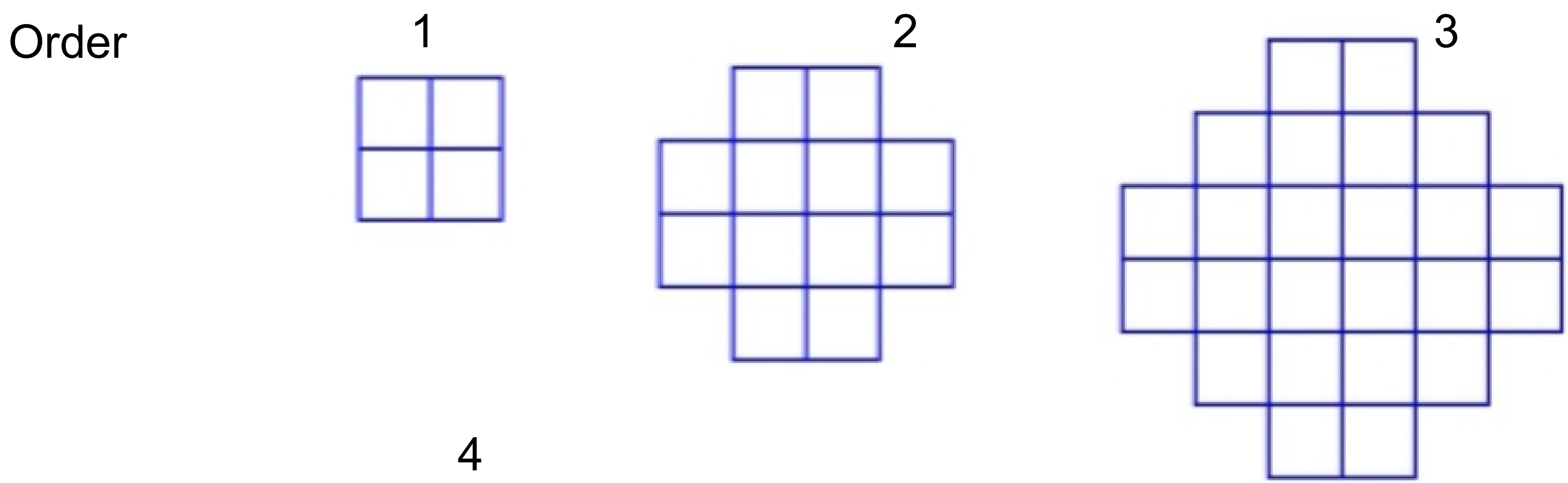
$L = 256$

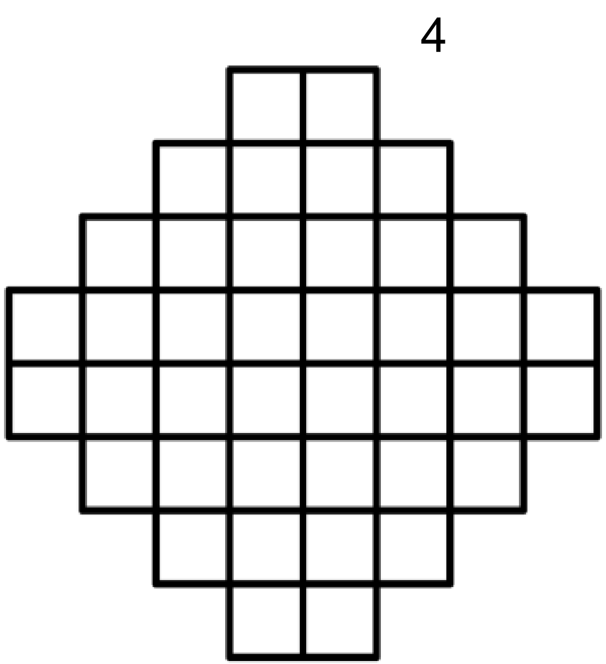
Themodynamic limit $\quad L \to \infty$

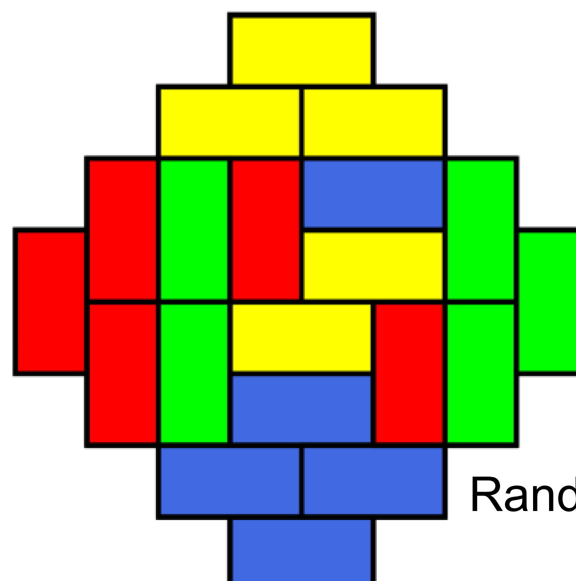
Number of configurations $\quad Z = e^{\frac{C}{\pi} L^2}$

$$C = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} \ldots \simeq 0.915965594$$

Aztec diamond



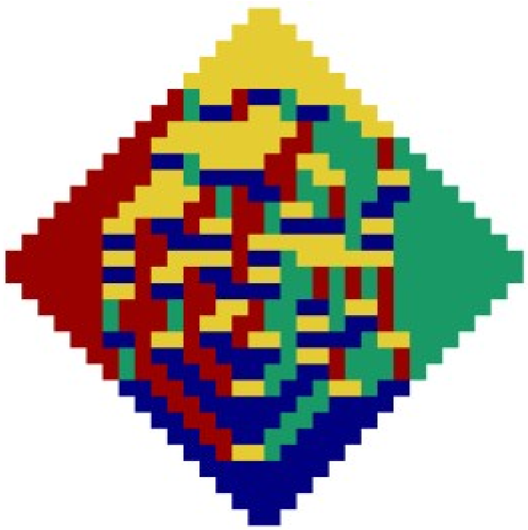




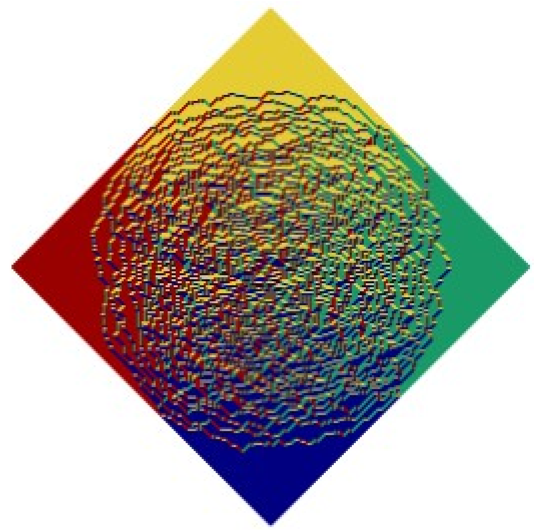
Random tiling of order 4 Aztec diamond

Aztec diamond in thermodynamic limit

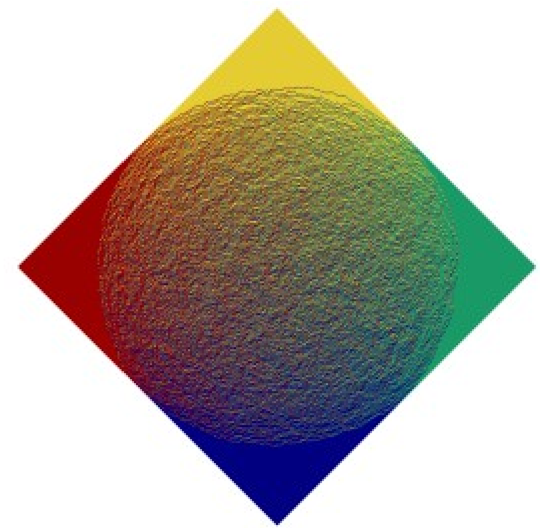
Stephane 2020



$L = 16$



$L = 128$



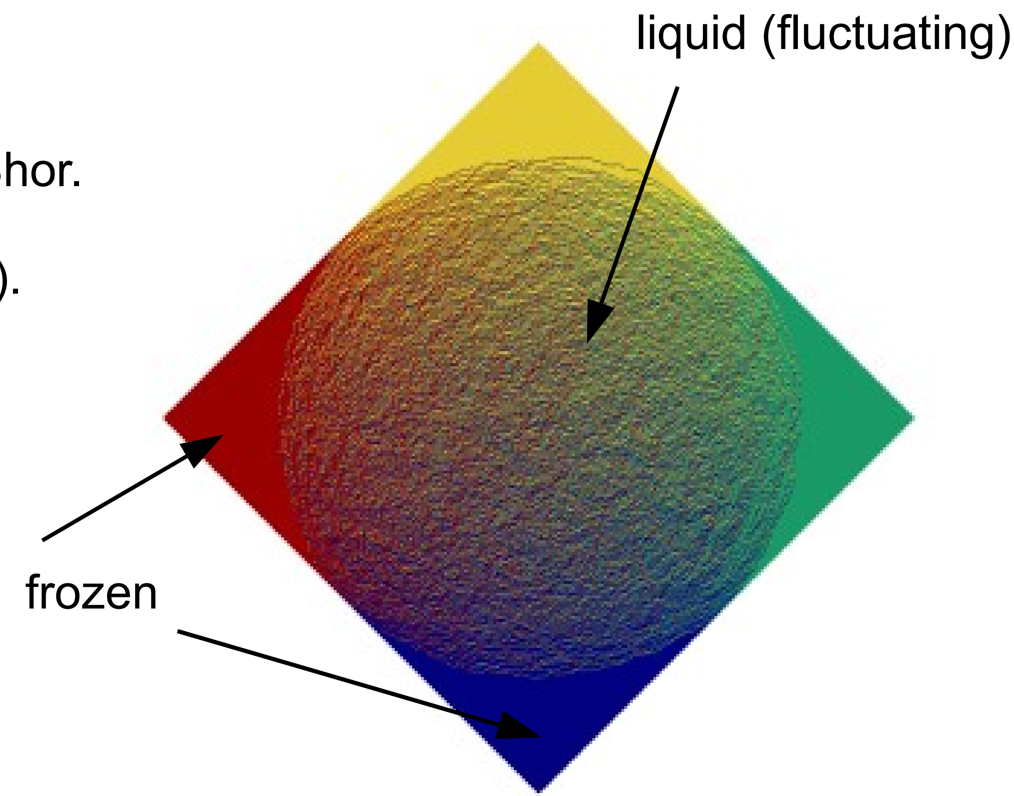
$L = 1024$

Number of configurations $$Z = 2^{L(L+1)/2}$$

Elkies, Kuperberg, Larsen and Propp,1992

Arctic circle theorem

Jockusch, William, James Propp, and Peter Shor. *"Random domino tilings and the arctic circle theorem."* arXiv preprint math/9801068 (1998).

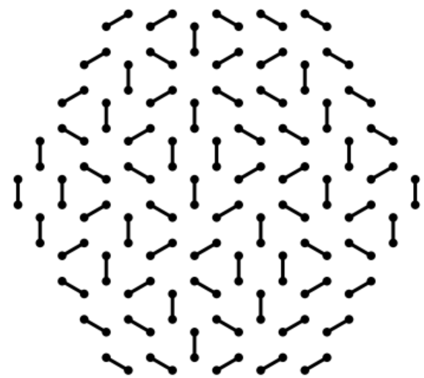


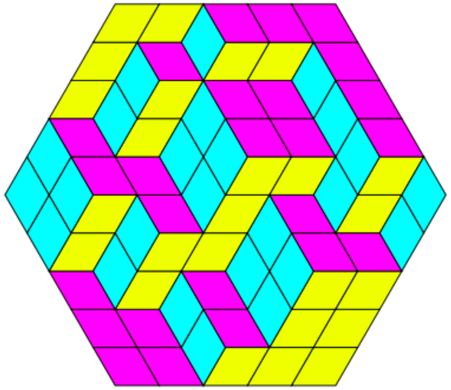
THEOREM 1 (the Arctic Circle Theorem): Fix $\epsilon > 0$. Then for all sufficiently large n, all but an ϵ fraction of the domino tilings of the Aztec diamond of order n will have a temperate zone whose boundary stays uniformly within distance ϵn of the inscribed circle.

Other examples

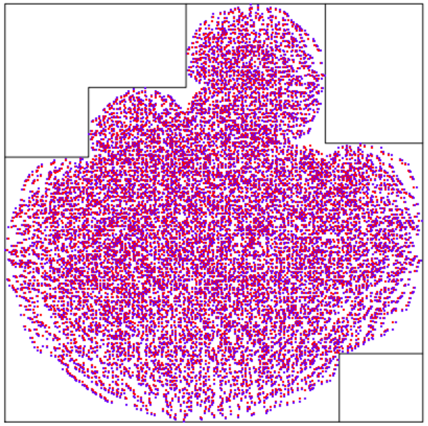
Hexagonal lattice

Belov, Enin, Nazarov 2018

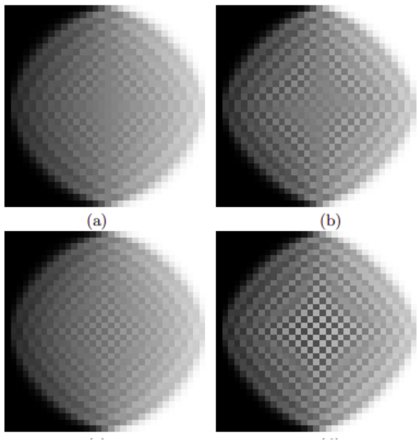




Interacting fermions (six-vertex model away from free-fermionic point)



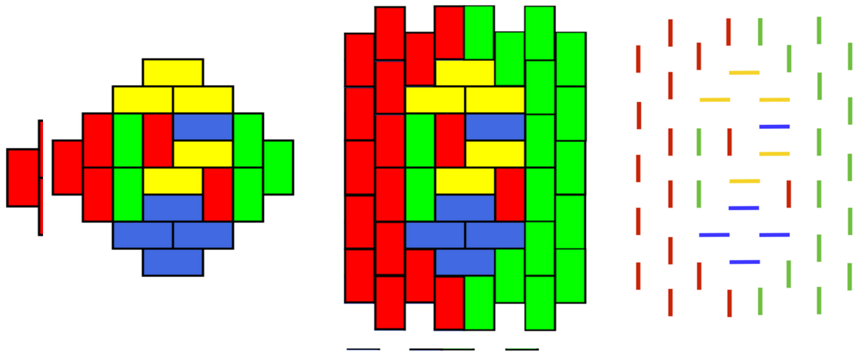
Colomo, Sportiello 2016



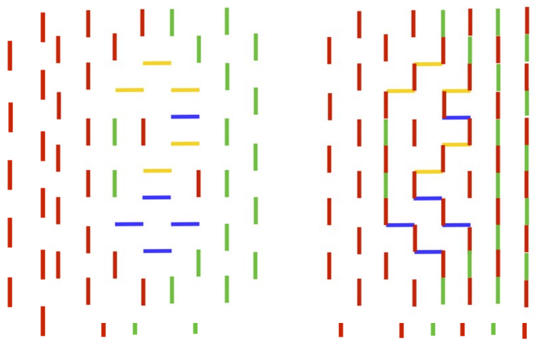
Syljuåsen, Zvonarev 2004

From tilings to particles

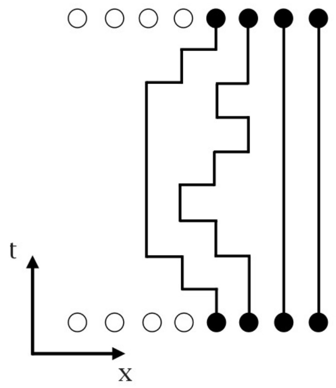
1. Convert tiles to dimers



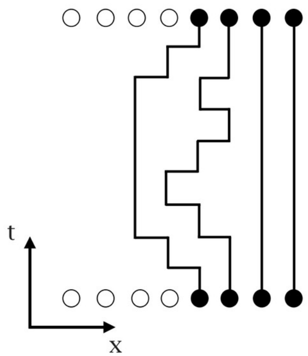
2. Superimpose obtained dimers with a reference state

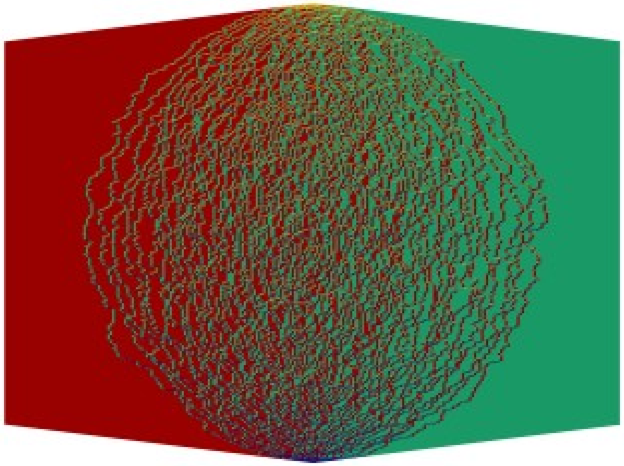


3. Obtain non-intersecting paths – world lines



From particles to fluid





Height field $h(x, \tau)$

Macroscopic hydro fields: $\rho(x, \tau) = -\partial_x h \quad j(x, \tau) = \rho v = \partial_\tau h$

$$\partial_\tau \rho + \partial_x j = 0$$

Hydrodynamic action

$$S[\rho, j] = \int \mathrm{d}\tau \mathrm{d}x \left(\frac{j^2}{2\rho} + E(\rho) \right)$$

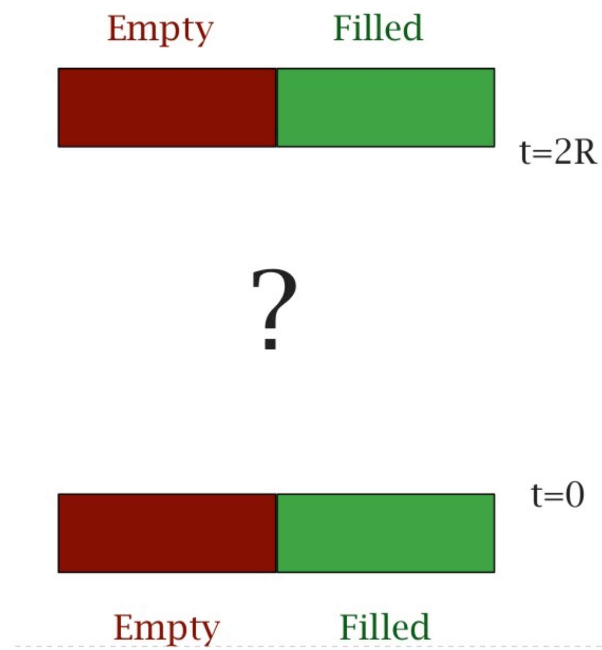
Internal energy

$$E(\rho) = \frac{\pi^2 \rho^3}{6}$$ - free fermions

Hydrodynamics and instantons

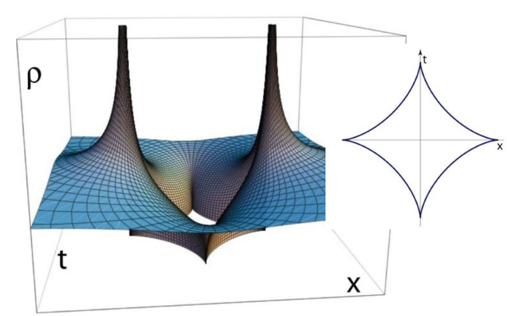
What is the optimal fluctuation of the gas in space and time so that at $\tau = 0$ and at $\tau = 2R$ the left half line is empty and the right one is filled?

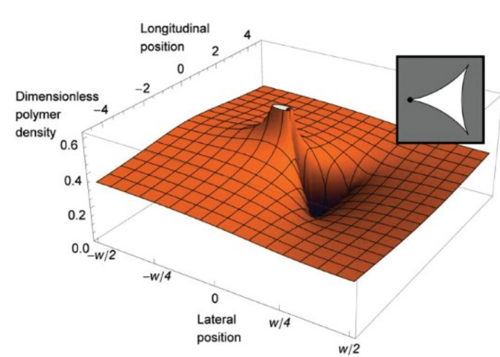
$$P \sim e^{-S_{\text{inst}}[\rho, j]}$$



Examples of hydrodynamic instantons:

- Arctic Circle (this talk)
- Emptiness in ground state of free fermions
- Pinned directed polymers





Equations of motion

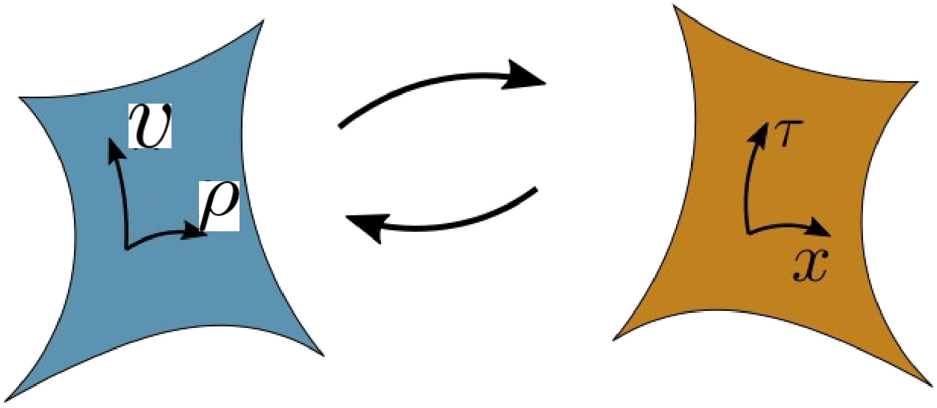
Continuity $\partial_\tau \rho + \partial_x(\rho v) = 0$

Euler $\partial_\tau v + v\partial_x v = \partial_x \frac{\pi^2\rho^2}{2}$

Complex Burgers
$$k, \bar{k} = \pi\rho \pm \mathrm{i}v$$
$$\mathrm{i}\partial_\tau k + k\partial_x k = 0$$

Solution - Complex characteristics $x + \mathrm{i}k\tau = g(k)$ - analytic function

For a given x, τ one can find $k, \bar{k} = \pi\rho \pm \mathrm{i}v$



Example of limiting shape in QM - emptiness formation probability

Imagine a Fermi gas on a line

Abanov 2003

What is the probability to have region of size R empty of particles

in the ground state?

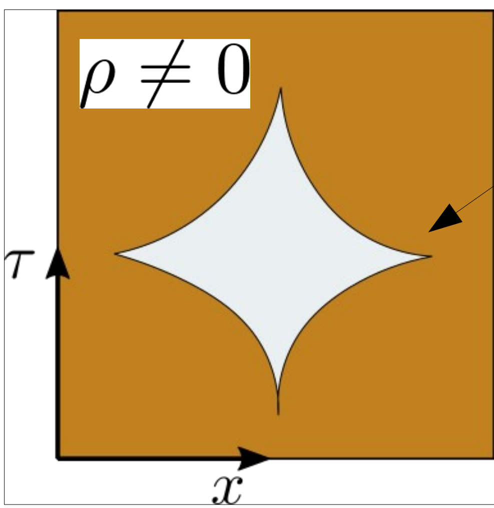
Hydrodynamic approach in imaginary time

$$P_R \sim e^{-S_{\rm opt}}$$

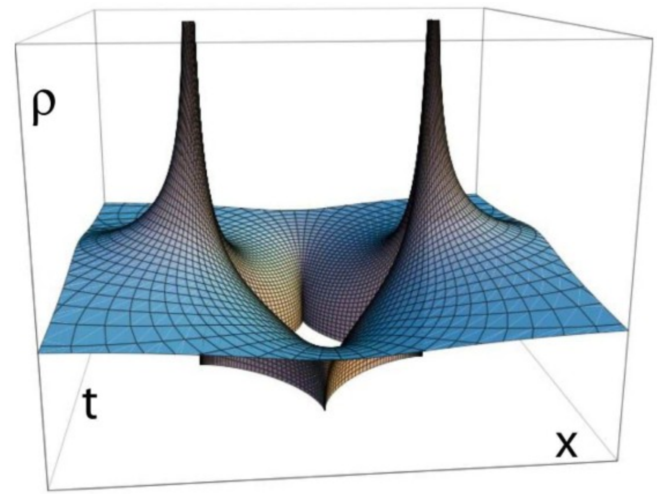
Optimal fluctuation – instanton of hydrodynamical fields

Optimal emptiness shape - astroid

$$x + ik\tau = g_{\mathrm{empt}}(k) = \frac{k}{\sqrt{k_F^2 - k^2}}$$



$$t \sim (R - x)^{2/3}$$



$$x^{2/3} + \tau^{2/3} = R^{2/3}$$

$$P_R \sim e^{-S_{\mathrm{opt}}}$$

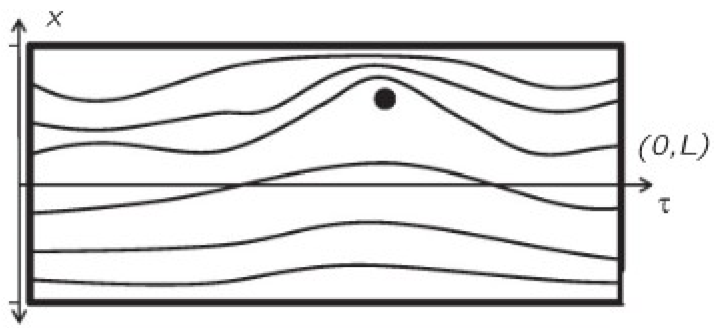
$$S_{\mathrm{opt}} = \frac{1}{2}(k_F R)^2 \sim$$ area in space-time

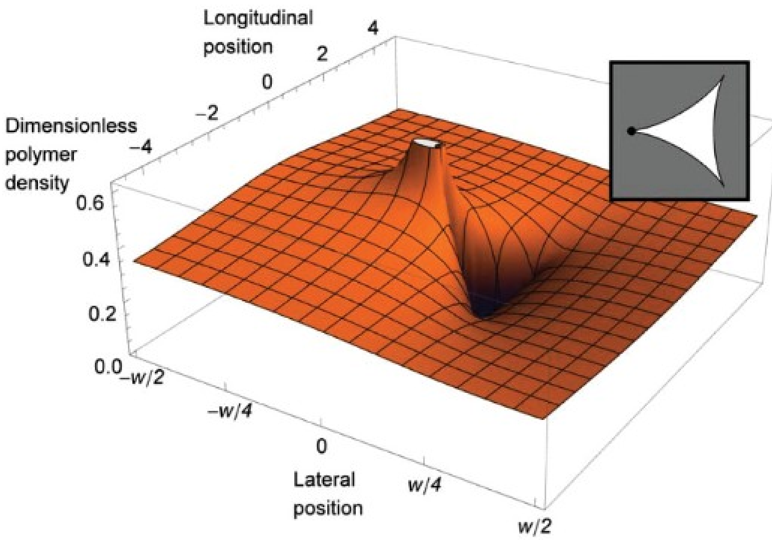
# of missing particles $k_F R \gg 1$

Directed Polymers

D. Zeb Rocklin, Shina Tan and Paul M. Goldbart, ‘12

Non-intersecting random paths + pins, barriers





Hydrodynamic instanton parametrised by

$$x + ik\tau = g_{\rm pin}(k) = x_{\rm s}\,\frac{1-a^2k^2}{1-k^2}$$

Question

How to find the analytic function $g(k) = x$

or the *spectral curve* $F_0(x, k) = 0$

which parametrises the hydro instanton solution ?

Both functions contain information about density/velocity of particles at $\tau = 0$

however the boundary conditions are imposed at $\tau = \pm R$

Back to particles - Quantum Mechanics

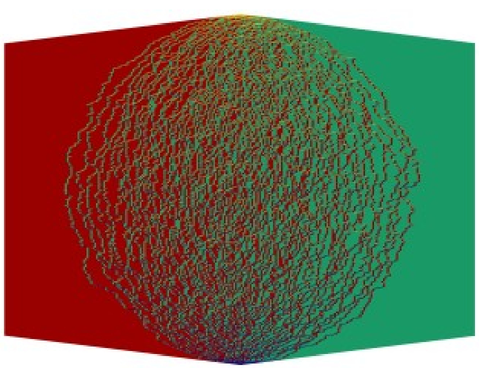
$$Z = \langle \Psi_0 | e^{-2RH} | \Psi_0 \rangle = \mathrm{Tr}\, e^{-2RH} | \Psi_0 \rangle \langle \Psi_0 |$$

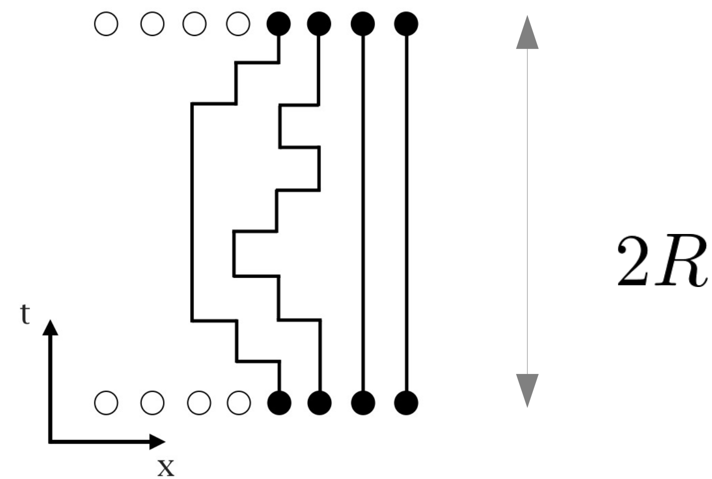
Free evolution (imaginary time) with tight-binding Hamiltonian

$$H = -\sum_j c^\dagger_{j+1} c_j + \mathrm{h.c.} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \varepsilon(k) c^\dagger(k) c(k)$$

$$\varepsilon(k) = -\cos(k)$$

$$\langle \Psi_0 | = \langle \ldots, 0, 0, 0, 1, 1, 1, \ldots |$$





$$| \Psi_0 \rangle = | \ldots, 0, 0, 0, 1, 1, 1, \ldots \rangle$$

Wick's theorem

boundary time slices

empty 1d lattice

$$Z_N = \langle \Psi | e^{-2RH} | \Psi \rangle$$

$$Z_N = \langle 0 | c_N(R) \ldots c_1(R) c_1^\dagger(-R) \ldots c_N^\dagger(-R) | 0 \rangle$$

$$Z_N = \det_{yx} \langle 0 | c_y(R) c_x^\dagger(-R) | 0 \rangle = \det_{yx} \int \frac{dk}{2\pi} e^{ik(x-y) - 2R\varepsilon(k)}$$

$$Z_N = \frac{1}{N!} \int \frac{d^N k}{(2\pi)^N} \left| \Delta(e^{ik}) \right|^2 e^{-2R \sum_l \varepsilon(k_l)}$$

$$\Delta(e^{ik}) = \prod_{i<j} (e^{ik_i} - e^{ik_j})$$

- Vandermonde determinant

Gross-Witten-Wadia model

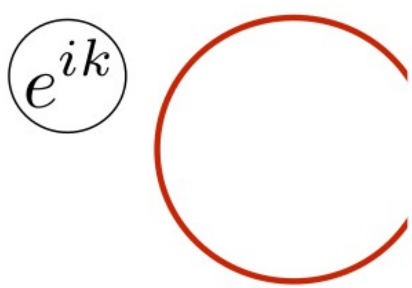
Gross, Witten, 1980, Wadia, 1980

Partition function for 2d U(N) lattice gauge theory was reduced to:

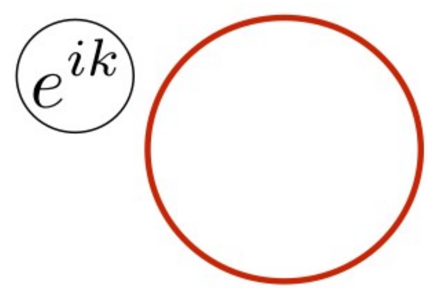
$$Z_N = \int dU \, \exp\left\{\frac{1}{\lambda}\mathrm{Tr}\,(U + U^\dagger)\right\} \qquad U = V\,\mathrm{diag}\left\{e^{ik_j}\right\}V^\dagger$$

$$= \frac{1}{N!}\int \frac{d^N k}{(2\pi)^N}|\Delta(e^{ik})|^2 e^{\frac{2N}{\lambda}\sum_l \cos(k_l)}$$

Third-order weak-strong coupling phase transition at the t'Hooft coupling $\lambda = 2$ in $N \to \infty$ limit



weak coupling $\lambda < 2$

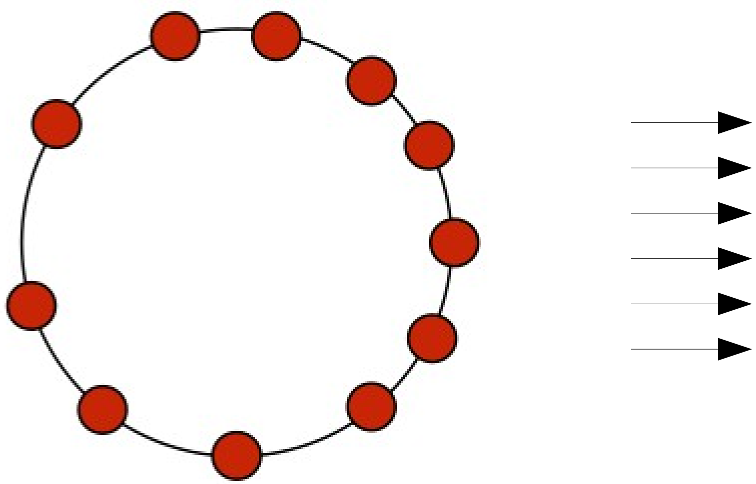


strong coupling $\lambda > 2$

Electrostatic interpretation

Charges on unit circle

$$Z_N = e^{-F_N} = \int d^N k \, e^{-E_N(k_1 \dots k_N)}$$



External electric field

$$\sim \frac{1}{\lambda} = \frac{R}{L}$$

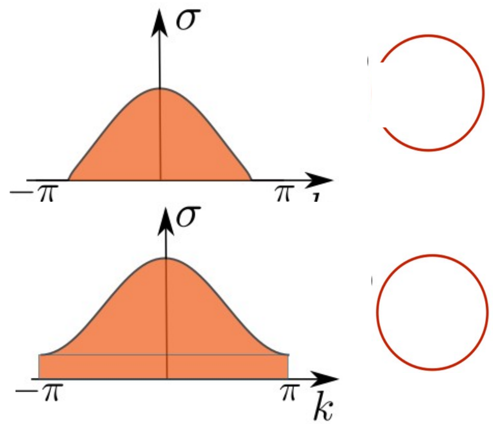
logarithmic repulsion external potential

$$E_N = -2 \sum_{i<j} \ln \left| e^{ik_i} - e^{ik_j} \right| - \sum_i \frac{2N}{\lambda} \cos k_i$$

Large N solution

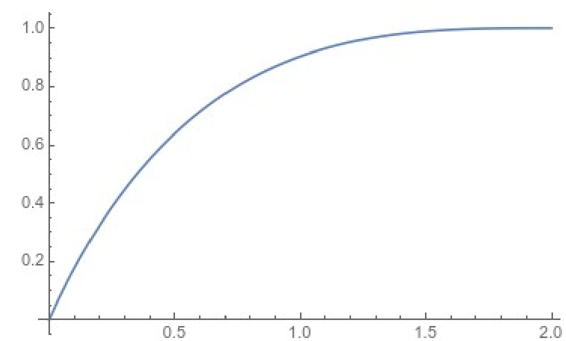
Density of charges: $$\sigma(k) = \frac{1}{N}\sum_i \delta(k - k_i)$$

$$2\pi\sigma(k) = \begin{cases} \frac{4}{\lambda}\cos\frac{k}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{k}{2}} & \lambda \leq 2 \\ 1 + \frac{2}{\lambda}\cos k & \lambda \geq 2 \end{cases}$$



Free energy

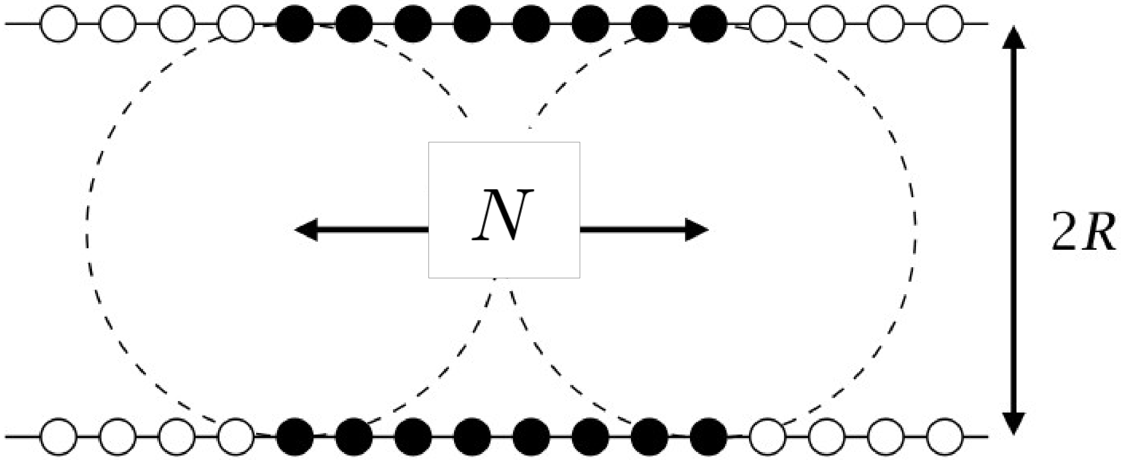
$$F_N = -R^2 \times \begin{cases} 2\lambda - \frac{3\lambda^2}{4} + \frac{\lambda^2}{2}\log\frac{\lambda}{2}\,, & \lambda < 2 \\ 1\,, & \lambda > 2 \end{cases}$$



$\frac{d^3 F_N}{d\lambda^3}$ is discontinuous at $\lambda = \frac{N}{R} = 2$

Gross, Witten, 1980
Wadia, 1980

Space-time picture of GWW transition



$$\lambda = \frac{N}{R}$$

Finite N ➡ two domain walls ➡ two arctic circles for left/right fermions

$\lambda > 2 \quad N > 2R$ left and right AC are independent $Z \sim e^{2\times R^2/2}$

$\lambda = 2 \quad N = 2R$ left and right AC touch each other

$\lambda < 2 \quad N < 2R$ Arctic circles overlap?

From k-space to real space

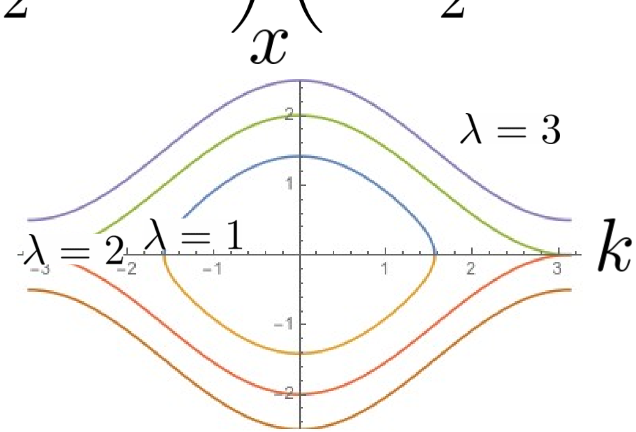
Coulomb Gas solution $\sigma(k)$ can be obtained from *loop equation*

$$F_0(\pi\lambda\sigma, k) = 0$$

Solution of the loop equation, or $x(k) = \pi\lambda\sigma(k)$
provides the desired $x - k$ relation parametrising hydro instantons

For double arctic circle

$$F_0(x,k) = \left(x - \frac{\lambda}{2} - \cos k\right)\left(x + \frac{\lambda}{2} + \cos k\right) - m^2(\lambda)$$

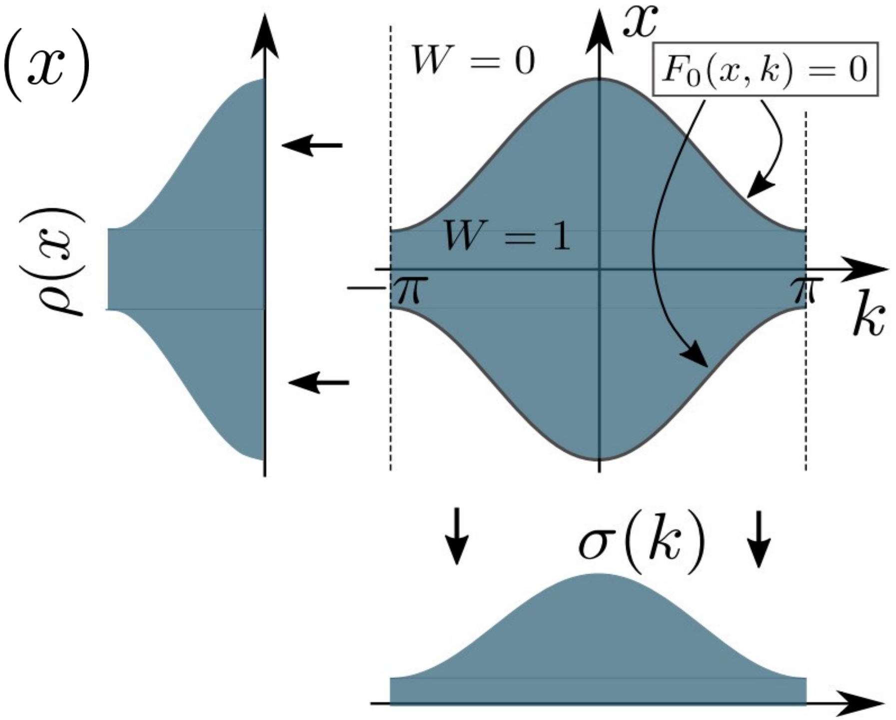


$$m = \begin{cases} 0\,, & \lambda > 2 \\ 1 - \frac{\lambda}{2}\,, & \lambda < 2 \end{cases}$$

Loop equation and semiclassical Wigner function

Semiclassically fermions occupy uniformly a region in the phase space.
Its boundaries are given

- implicitly by $F_0(x,k) = 0$
- or explicitly by $x_\pm(k) \quad k_\pm(x)$



$$\sigma(k) = x_+(k) - x_-(k)$$

$$2\pi\rho(x) = k_+(x) - k_-(x)$$

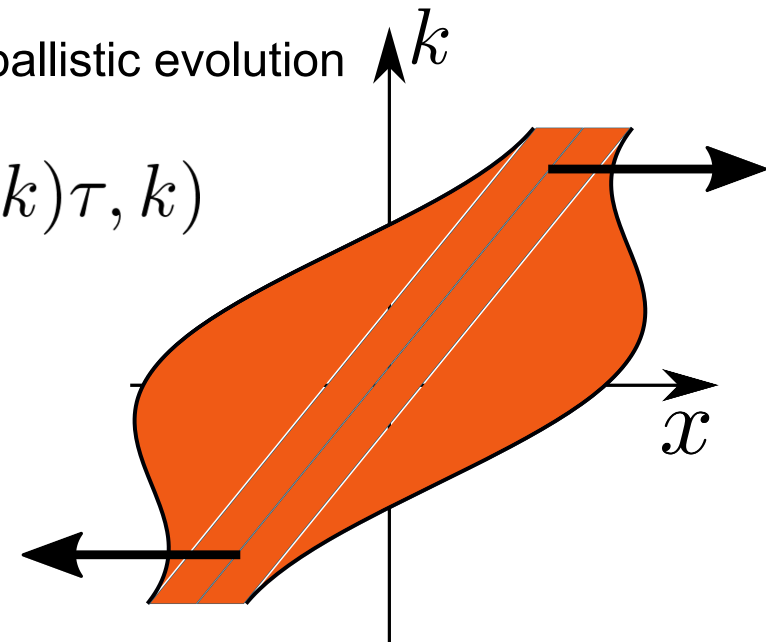
Ballistic evolution in imaginary time

Up to now we were working on $\tau = 0$ slice with real x, k due to time-reversal symmetry

Extension to other time slices is straightforward – ballistic evolution

$$F_0(x,k) \to F_\tau(x,k) = F_0(x - i\varepsilon'(k)\tau, k)$$

- need to consider complex values of x, k

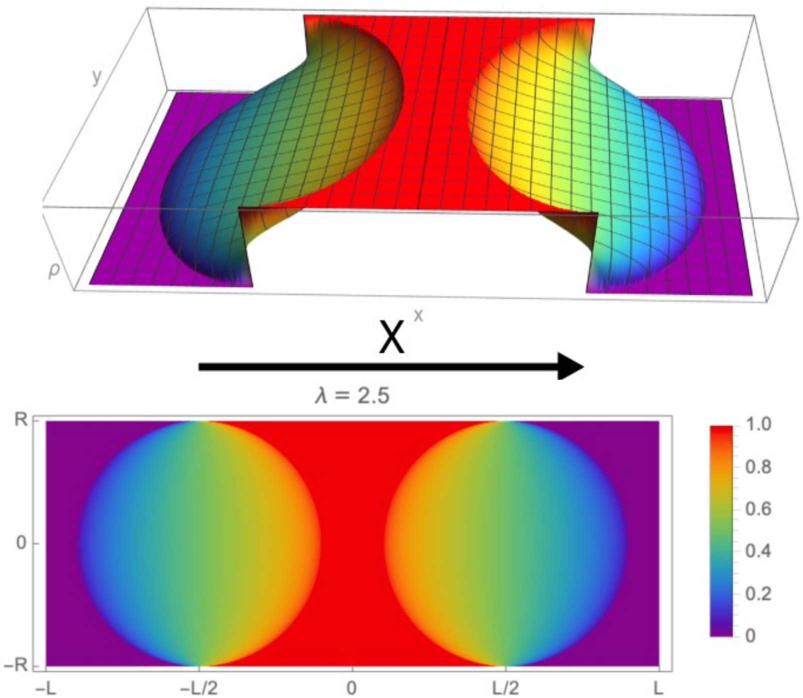


Solving $F_\tau(x,k) = 0$ gives optimal density and velocity profiles

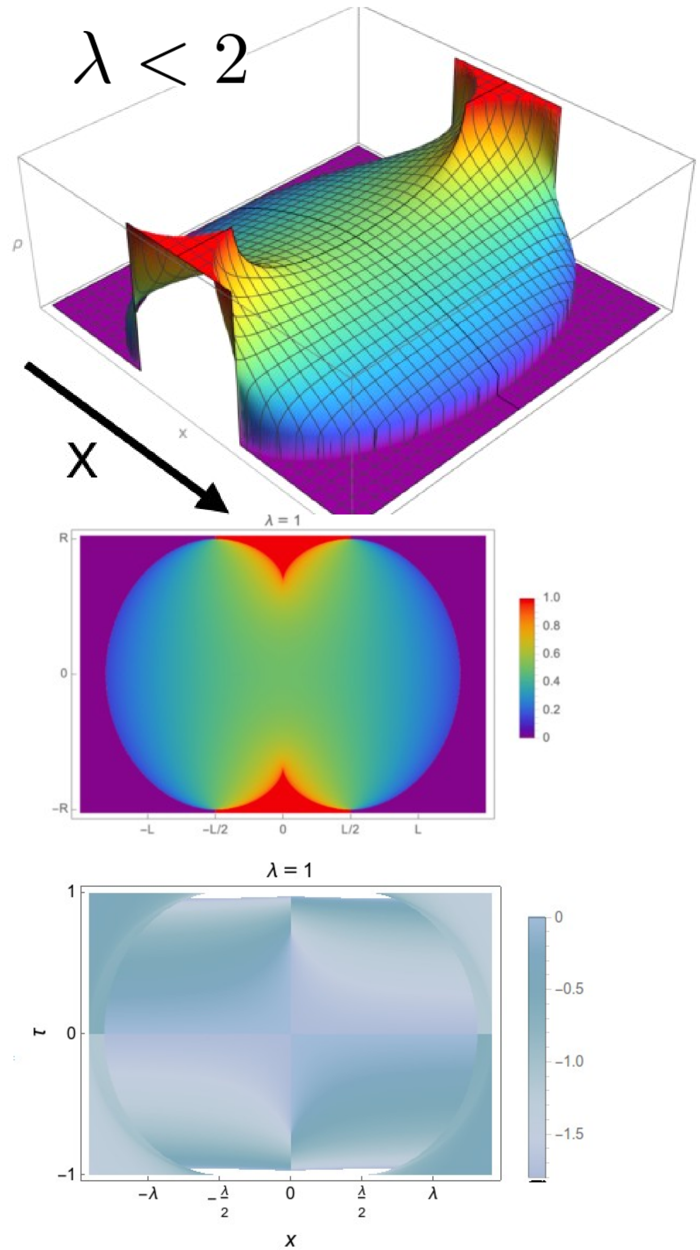
$$k_\pm(x) = \pi\rho(x) + iv(x)$$

Pictures

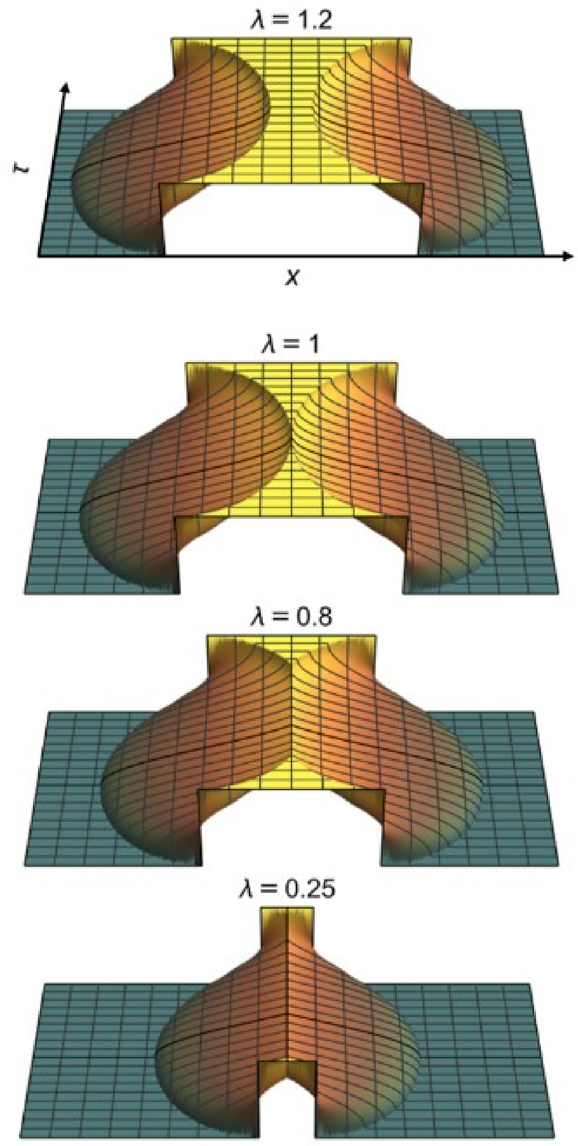
$\lambda > 2$



$\lambda < 2$



Pictures



Frozen boundary

Equation $F_\tau(x,k) = 0$ has four solutions $k(x,\tau)$

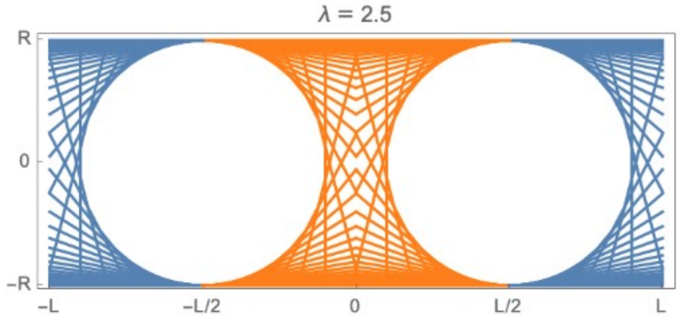
They coalesce pairwise at $k(x,\tau) = iv(x,\tau)$, or $\pi + iv(x,\tau)$

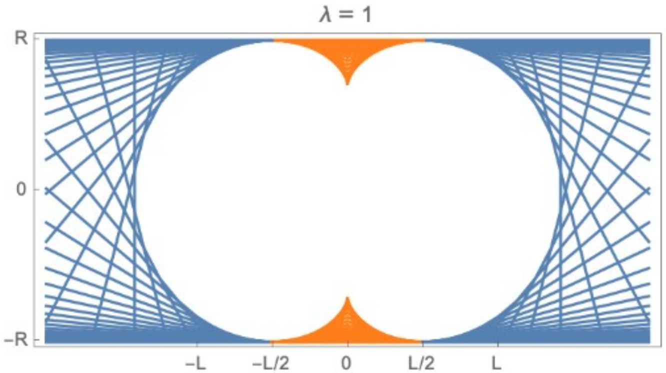
corresponding points (x,τ) are empty/full *frozen boundaries*
obtained from imposing additional condition

$$\partial_k F_\tau(x,k) = 0$$

Envelopes of straight line families (caustics)

$$x = G(v) \pm \tau \sinh v \qquad \tau = G'(v) \qquad G(v) = g(iv)$$



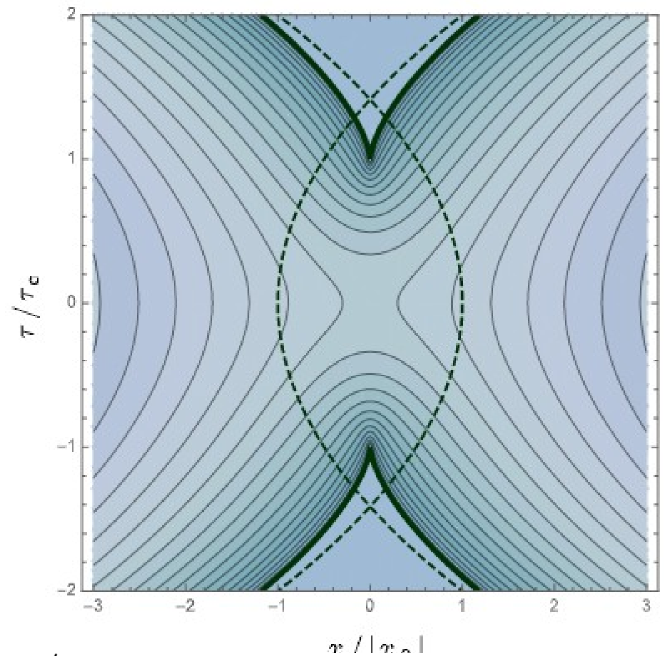


Near the merger transition

Critical central region is described by universal function

$$F_\tau(x,k) = (x - i\tau k)^2 + \theta(-x_0)x_0^2 - \left(x_0 + \frac{k^2}{2g}\right)^2$$

depending on 2 parameters x_0, g



e.g. the right boundary is

Separated phase $x_0 > 0$ $\qquad x(\tau) = x_0 + \frac{g\tau^2}{2}$

Merged phase $x_0 < 0$ $\qquad x(\tau) = \frac{8|x_0|}{3\sqrt{6}}\left(\left|\frac{\tau}{\tau_c}\right| - 1\right)^{3/2} \qquad 0 < \tau - \tau_c \ll \tau_c$

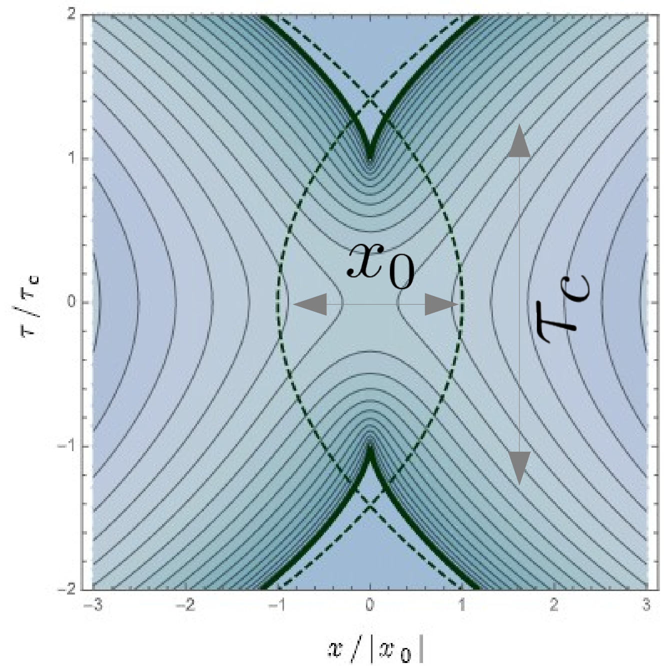
$$x(\tau) = -|x_0| + \frac{g\tau^2}{2}, \qquad |\tau - \tau_c| \gg \tau_c$$

$$\tau_c = \sqrt{|x_0|/g}$$

Third order phase transition

Density of holes below transition

$$\delta\rho(0,0) = \frac{2\sqrt{g|x_0|}}{\pi}$$



Contributes to energy density (free fermions)

$$\delta E = \delta\rho^3$$

Action cost

$$\delta S \sim \delta\tau\delta x\delta E = (|x_0|^{3/2}/\sqrt{g})g^{3/2}|x_0|^{3/2} = g|x_0|^3$$

In interacting model free fermionic decription is valid for small densities!

Fluctuation length scale $\ell \sim g^{-1/3}$

In GWW model (free fermions) $g = 1/R$ and $\ell = R^{1/3}$

Transitions in Real-time dynamics

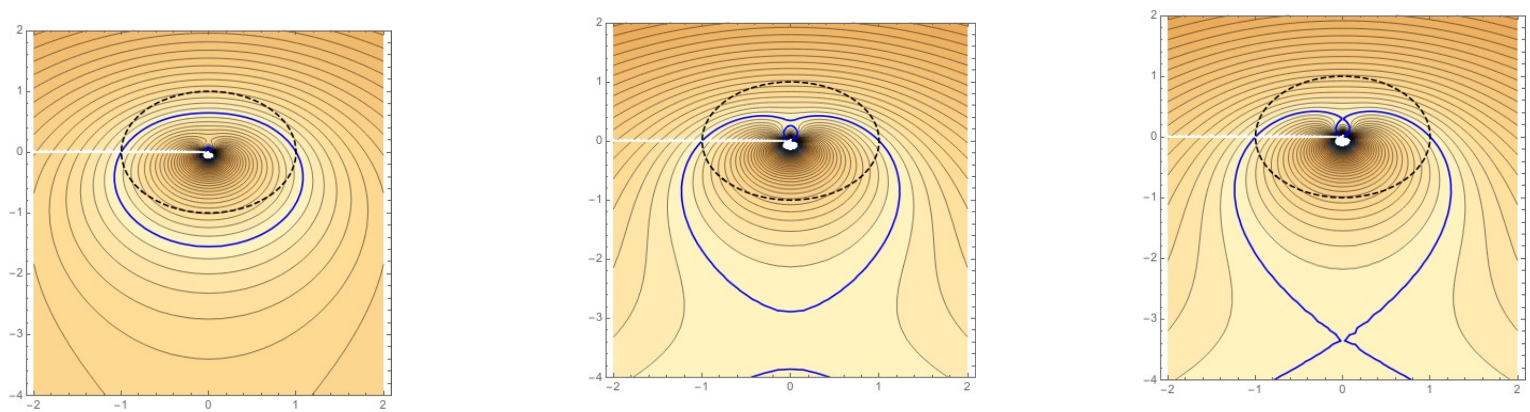
Work in progress with Yasser Bezzaz

Loschmidt echo $R \to iR$ *P L Krapivsky, J M Luck and K Mallick, ‘18*

$$\left|\langle N|e^{-2iRH}|N\rangle\right|^2 \sim \begin{array}{ll} e^{-R^2}, & R \gg N \\ R^{-N^2}, & R \gg N \end{array}$$

Coulomb Gas approach is still valid, but charges leave unit circle

Phase transition at critical $\lambda = N/R = \lambda_c \simeq 3.018\ldots$



Conclusions and open problems

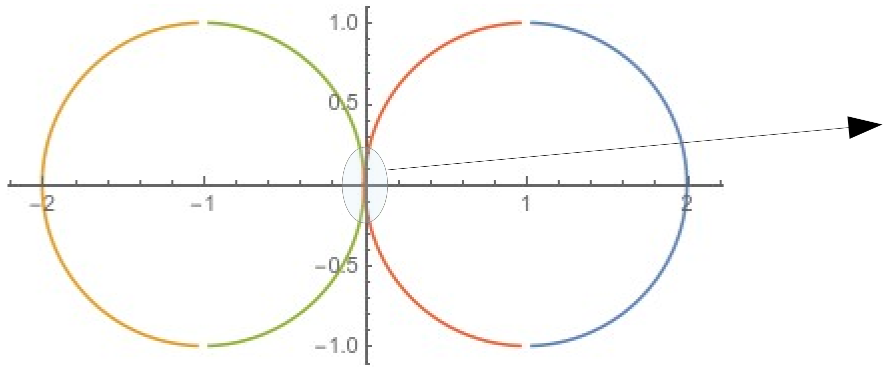
- Limit Shape Phenomena occur in many statistical/random problems
- We map two Arctic Circle problem onto Gross-Witten-Wadia model.
- Third order phase transition in GWW model can be interpreted as melting of the frozen region between the two Arctic Circles and their merger.
- It is conjectured that the transition is of the third order even in the presence of interactions (protected by P and T symmetries)

- More complicated dispersions
- Hydrodynamic instanton approach can be generalised to interacting models (XXZ,six vertex,...)
- Real time dynamics from $\tau \to it$: Quantum Quenches, Quantum Information (projective and weak measurements), Floquet evolution and Time Crystals….

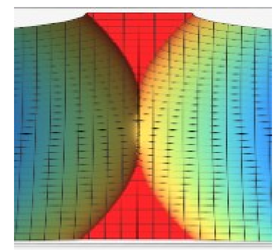
Շնորհակալություն

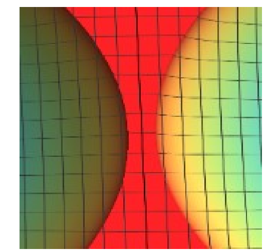
Near the melting transition

$\lambda \sim 2$

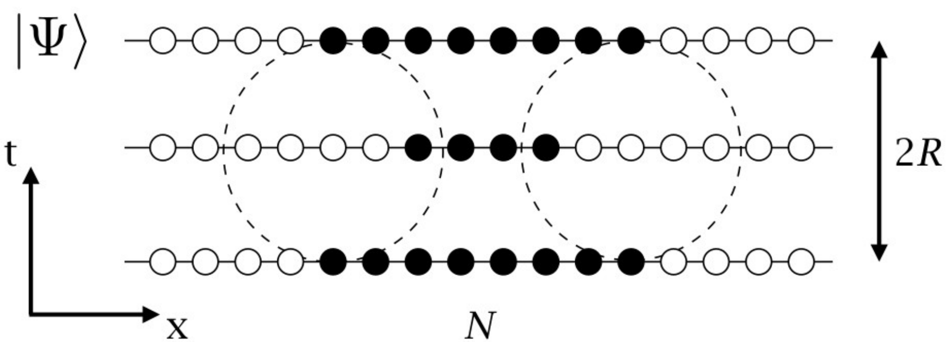


$\lambda = 2 - a/R^{1/3}$





We calculate “fullness” or “iceberg” formation probability by considering optimal configurations with additional BC



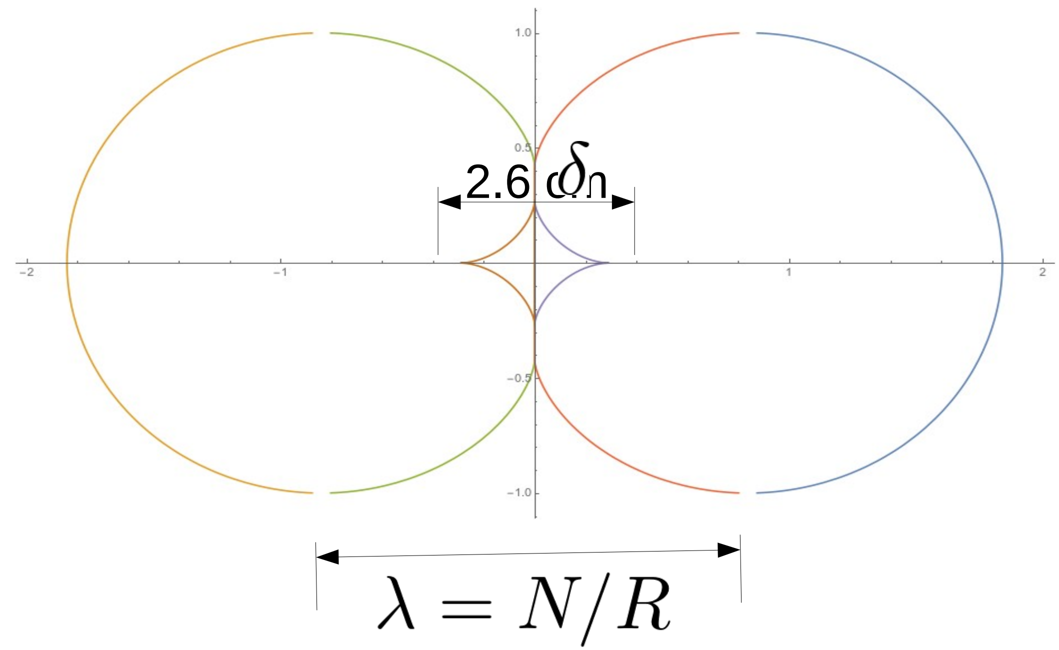
Fullness Formation Probability

$$\lambda = 2 - a/R^{2/3}$$

$$\delta = xR^{-2/3}$$

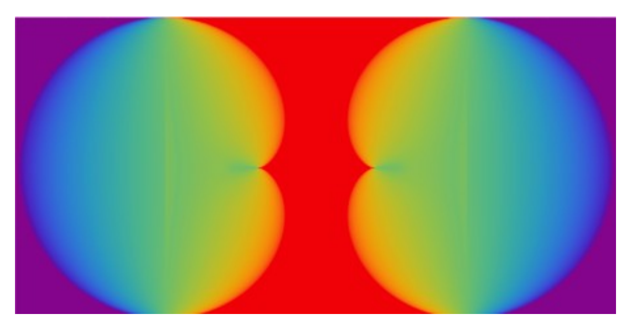
$$P(x,a) \sim e^{-S_{\rm opt}(x,a)}$$

$$S_{\rm opt} = x^2 + \frac{2}{3}\left(\frac{x}{a}\right)^3$$



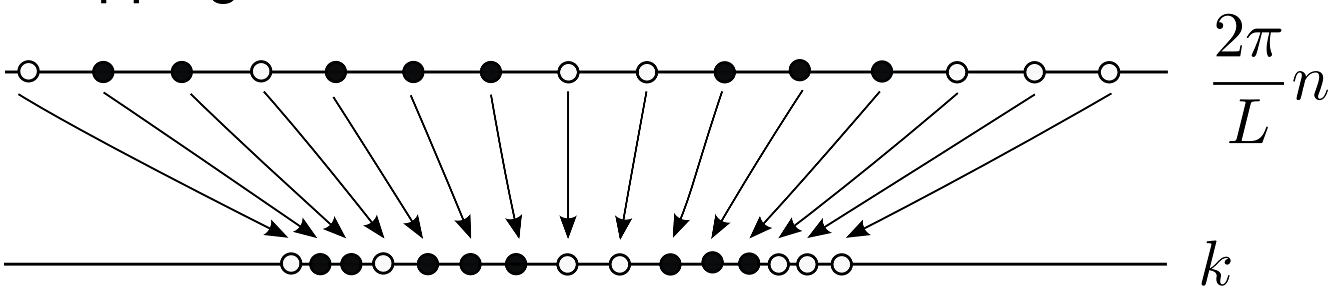
Gaussian decay (fullness in the middle of fluctuating region $x \ll a$

Asymptotics of Tracy-Widom distribution $x \gg a$

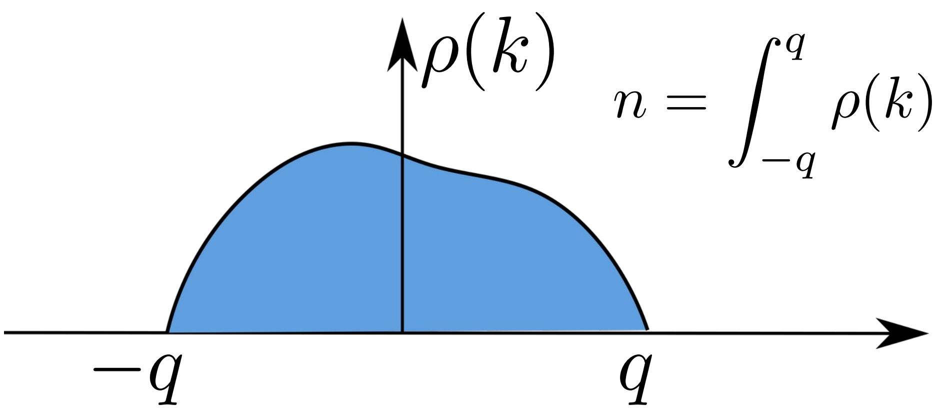


Bethe Ansatz Solution (LL)

Mapping from free fermions



Quasi-momentum (rapidity) distribution



$$n = \int_{-q}^{q} \rho(k) \mathrm{d}k$$

Total momentum

$$P = \sum_{j=1}^{N} k_j = \int_{-q}^{q} k\rho(k)\, \mathrm{d}k$$

Total energy

$$\varepsilon = \sum_{j=1}^{N} \frac{k_j^2}{2m} = \int_{-q}^{q} \frac{k^2}{2m} \rho(k)\, \mathrm{d}k$$